Lecture No. 23

Measure and Integration

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$f \in L_1[a,b]$

$$f = f^+ - f^-$$

$$f \in L_1[a,b] \iff f^+, f^- \in L_1[a,b]$$

If $\exists\ g_1 \in C[a,b]$, $g_2 \in C[a,b]$

such that

$$\| f^+ - g_1 \|_1 < \varepsilon$$

$$\| f^- - g_1 \|_1 < \varepsilon$$

$$\Longrightarrow \quad \| f - \underbrace{(g_1 - g_2)}_{= g} \|_1 = \| f^+ - f^- - (g_1 - g_2) \|$$

$$\leq \| f^+ - g_1 \|_1 + \| f^- - g_2 \|$$

$$< \varepsilon + \varepsilon$$

Given $f \geqslant 0$, $f \in L_1[a,b]$

$f \geqslant 0$, f mble $\Rightarrow$ $\exists \{s_n\}_{n \geqslant 1}$ of non-negative simple measurable fns, $s_n \uparrow f$

$$0 \leq s_n \leq f$$

$$\Rightarrow s_n \in L_1[a,b]$$

<u>Also</u>

$$\int f \, d\mu = \lim_{n \to \infty} \int s_n \, d\mu$$

$$\|f - s_n\|_1 = \int |f - s_n| \, d\mu = \underbrace{\int f \, d\mu - \int s_n \, d\mu}_{\longrightarrow 0}$$

$\Rightarrow \forall \varepsilon > 0$, $\exists n_0$ s.t $\|f - s_{n_0}\| < \varepsilon$.

$s \in L_1[a,b]$.

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad \left\{ \begin{array}{l} a_i \geq 0 \\ A_i \subseteq [a,b] \\ A_i \cap A_j = \phi \\ \bigcup_{i=1}^{n} A_i = [a,b] \end{array} \right.$$

To show $\exists\ g \in C[a,b]$

$\|s - g\|_1 < \varepsilon$?

Enough to show $\forall\ A \subseteq [a,b],\ A \in \mathcal{L}$

$\exists\ g \in C[a,b]$ such that

$$\|\chi_A - g\|_1 < \varepsilon.$$

Because of time

$\forall\, i$, find $g_i \in C[a,b]$ such that

$$\| \chi_{A_i} - g_i \|_1 < \varepsilon \quad \forall\, i = 1, 2 \ldots n$$

$$\Rightarrow \quad g := \sum_{i=1}^{n} a_i g_i \in C[a,b]$$

and

$$\Big\| \sum_{i=1}^{n} a_i \chi_{A_i} - g \Big\|_1 \le \sum_{i=1}^{n} |a_i| \, \| \chi_{A_i} - g_i \|_1 < \sum_{i=1}^{n} |a_i| \, \varepsilon$$

Let $A \subseteq [a,b]$, $A \in \mathcal{L}$.

<u>To show</u> $\exists\, g \in C[a,b]$ s.t.

$$\|\chi_A - g\|_1 < \varepsilon \quad ?$$

<u>Note</u>

$$\lambda(A) = \lambda^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(I_j) \;\middle|\; A \subseteq \bigcup_{j=1}^{n} I_j,\ I_i \cap I_j = \phi \right\}$$

$\Longrightarrow \forall\, \varepsilon > 0$, $\exists\, I_n$ intervals, $n = 1, 2, \ldots$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n, \quad I_n \cap I_m = \phi.$$

and

$$\lambda^*(A) + \varepsilon/2 > \sum_{n=1}^{\infty} \lambda(I_n)$$

$$\sum_{n=1}^{\infty} \lambda(I_n) < +\infty$$ 7

We can select n_0 such that

$$\sum_{n_0+1}^{\infty} \lambda(I_n) < \varepsilon/2 \quad \text{——} \quad ①$$

Then

$$A \setminus \bigcup_{n=1}^{n_0} I_n \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \setminus \left(\bigcup_{n=1}^{n_0} I_n\right)$$

$$= \bigcup_{n=n_0+1}^{\infty} I_n$$

$$\Longrightarrow \quad \lambda\left(A \setminus \bigcup_{n=1}^{n_0} I_n\right) < \sum_{n=n_0+1}^{\infty} \lambda(I_n) < \varepsilon/2$$

Also

$$\bigcup_{n=1}^{n_0} I_n \setminus A \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \setminus A$$

$$\lambda\Big(\Big(\bigcup_{n=1}^{\infty} I_n\Big)\setminus A\Big) \leq \lambda\Big(\bigcup_{n=1}^{\infty} I_n\Big) - \lambda(A)$$

$$\leq \sum_{n=1}^{\infty}\lambda(I_n) - \lambda(A)$$

$$< \varepsilon/2$$

$$\Longrightarrow \quad \lambda\Big(A \,\Delta \Big(\bigcup_{n=1}^{n_0} I_n\Big)\Big) < 2\,\varepsilon/2 = \varepsilon$$

" If $A \subseteq [a,b]$, given $\varepsilon > 0$, $\exists$ disjoint intervals $I_1, \ldots, I_{n_0}$ such that $\lambda\Big(A \,\Delta \Big(\bigcup_{n=1}^{n_0} I_n\Big)\Big) < \varepsilon$ "

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$$\int \left| \chi_A - \chi_{\bigcup_{n=1}^{n_0} I_n} \right| < \varepsilon$$

$$\left(\chi_{A \triangle B} = |\chi_A - \chi_B| \right)$$

$$\|$$

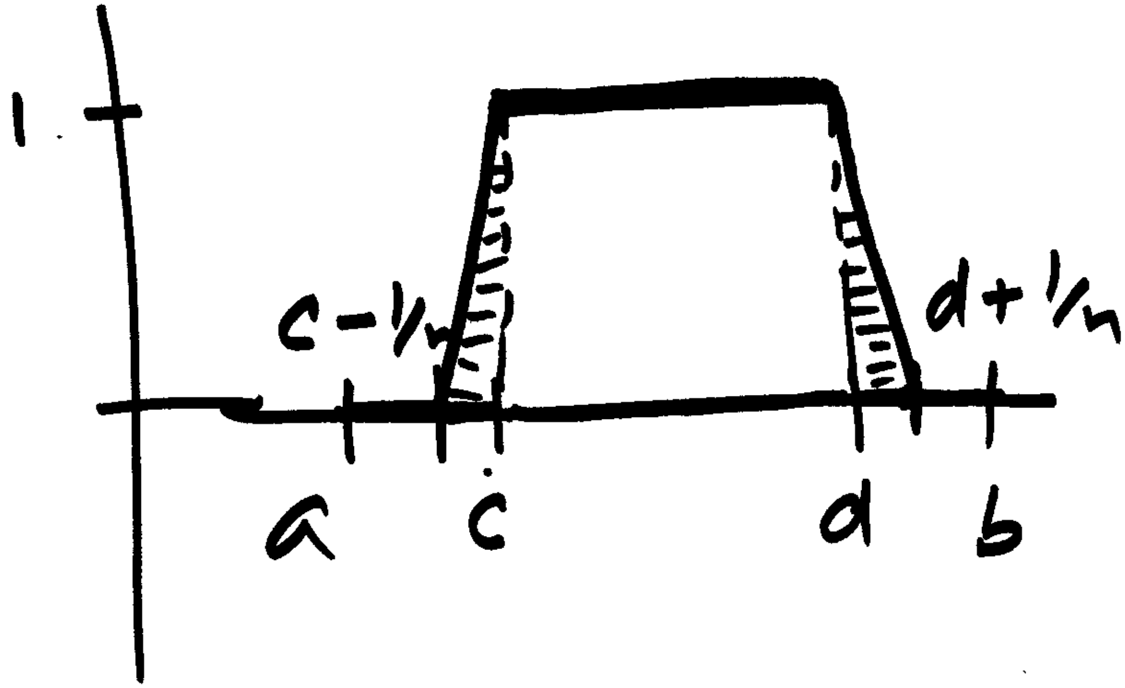
$$\int \left| \chi_A - \sum_{n=1}^{n_0} \chi_{I_n} \right| d\lambda < \varepsilon$$

$$\|$$

$$\left\| \chi_A - \sum_{n=1}^{n_0} \chi_{I_n} \right\|_1 < \varepsilon$$

Further if $I \subseteq [a,b]$, then

$\exists$ $g \in C[a,b]$, $\| \chi_I - g \| < \varepsilon$



$$g_n : \begin{cases} 0 & \text{in } (a, c-\frac{1}{n}) \\ \text{line} & (c-\frac{1}{n}, c] \\ 1 & (c,d) \\ \text{line} & (d, d+1/n) \\ 0 & \underline{\qquad\qquad} \end{cases}$$

$$\int |\chi_I - g_n| \, d\lambda = \frac{2}{n} \times \frac{1}{2} = \frac{1}{n} \xrightarrow[\text{as } n \to \infty]{} 0$$

$$\int f_h(x)\, d\lambda(x) = \int f(x)\, d\lambda(x)$$

$\forall \ f \in L_1(\mathbb{R})$.

Step 1 Show true for $f \geq 0$

Step 2 Show true for $s \in L_1$, $s \geq 0$ nonnegative simple.

Step 3 Show true for χ_A

i.e. $\int \chi_A(x+h)\, d\lambda(x) = \int \chi_A\, d\lambda$

$\lambda(A+h) = \lambda(A) \ \forall h.$

$$\int f(kx)\,d\lambda(x) = |k| \int f(x)\,d\lambda(x)$$

$$\lambda(k\cdot E) = |k|\,\lambda(E), \quad E \in \mathcal{L}_{\mathbb{R}}$$